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# A soluble one dimensional model of an interacting Fermi gas: I†

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MS received 5 May 1971, in revised form 19 November 1971

**Abstract.** We study a one dimensional model system of fermions which proves to be exactly soluble in the thermodynamic limit. It is shown that the thermodynamic functions for the model system are the same as for a system of noninteracting fermions in a harmonic oscillator well. As a special case of the model system, with a particular choice for the coupling constant, it reduces to a particular two-species system of fermions interacting via pairwise harmonic forces. For this case of the two-species system the free energy per particle, the pressures, the specific heat at constant volume and the isothermal compressibility are calculated in the canonical ensemble in closed form. It is shown that this system does not undergo a phase transition.

## 1. Introduction

In recent years considerable effort has been expended in the study of one dimensional models for many body systems (Lieb and Mattis 1966). The reason for this interest is twofold. First, one dimensional problems are easier to solve than the analogous three dimensional problem. Solution of one dimensional problems may, therefore, point the way towards a better understanding of the domain of validity and the limitations of those approximations employed in the study of three dimensional systems. Secondly, such physical systems as chain molecules and some organic macromolecules may be described by one dimensional models and therefore studying such models will lead to a better understanding of these physical systems.

In § 2 we present a model Hamiltonian which is treated exactly, based on the double-time Green function technique. In particular, the thermodynamic properties of the system are determined. It is shown that in the thermodynamic limit, the interaction term in the model Hamiltonian does not contribute to the partition function of the canonical ensemble and the system is equivalent to a set of noninteracting fermions in an attractive harmonic oscillator well. In § 3 we evaluate without approximation the canonical partition function of the resulting noninteracting Fermi gas. In a suitable limit which is consistent with the thermodynamic limit, the model Hamiltonian reduces to the Hamiltonian representing a one dimensional system of  $2N$  point particles,  $N$  of which have a mass  $m$  ('pseudoelectrons'), obey Fermi-Dirac statistics and interact with each other via a pairwise repulsive harmonic potential. The other  $N$  point particles ('pseudoions') are assumed to have an infinite mass, interact with each other via the repulsive harmonic potential and interact with the pseudoelectrons via an attractive

† Work supported in part by the National Science Foundation and the US Air Force Office of Scientific Research.

potential of the same form. In § 4 we evaluate the canonical partition function for this case. In § 5 we calculate the thermodynamic functions for the pseudoelectron system in the quantum region.

**2. The free energy of the system—an exact calculation**

Consider the following model Hamiltonian for a one dimensional system of particles obeying Fermi–Dirac statistics :

$$\begin{aligned}
 H_{\lambda\delta} &= H_{\lambda 0} + H_1 \\
 H_{\lambda 0} &= \sum_{i=1}^N \frac{p_i^2}{2m} + (1-\lambda)\frac{1}{2}m\omega^2 \sum_{i=1}^N x_i^2 \\
 H_1 &= \delta \frac{m\omega^2}{2N} \left( \sum_{i=1}^N x_i \right)^2
 \end{aligned}
 \tag{1}$$

where  $\delta$  is a constant which will later be set equal to unity. We will show that in the thermodynamic limit and for  $-1 < \lambda < 1$  the free energy per particle  $f_{\lambda,\delta}$  corresponding to  $H_{\lambda\delta}$  equals the free energy per particle  $f_{\lambda 0}$  corresponding to the  $H_{\lambda 0}$  system of non-interacting fermions in a harmonic oscillator well.

In the limit  $\lambda \rightarrow 1 - 0$  the Hamiltonian (1) reduces to

$$\begin{aligned}
 H &= H_0 + H_1 \\
 H_0 &= \sum_{i=1}^N \frac{p_i^2}{2m} \\
 H_1 &= \delta \frac{m\omega^2}{2N} \left( \sum_{i=1}^N x_i \right)^2.
 \end{aligned}
 \tag{2}$$

The Hamiltonian (2) represents a one dimensional system consisting of  $2N$  point particles  $N$  of which have a mass  $m$  (‘pseudoelectrons’) which interact with each other via a pairwise repulsive harmonic potential, and  $N$  particles of infinite mass (‘pseudoions’) interacting with each other via a pairwise repulsive harmonic potential and interacting with the pseudoelectrons via an attractive potential of the same form. The Hamiltonian for such a system is

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} - \frac{1}{4}\alpha \sum_{i,j=1}^N (x_i - x_j)^2 + \frac{1}{2}\alpha \sum_{i,j=1}^N (x_i - X_j)^2 - \frac{1}{4}\alpha \sum_{i,j=1}^N (X_i - X_j)^2 \tag{3}$$

where  $x_i$  is the position of the  $i$ th pseudoelectron,  $X_i$  the position of the  $i$ th pseudoion and  $\alpha$  is a coupling constant which depends on the size of the system. We shall assume that each species of  $N$  particles separately obeys Fermi–Dirac statistics. The transformation

$$x_i \rightarrow x_i + N^{-1} \sum_{j=1}^N X_j \tag{4}$$

corresponding to measuring the coordinate  $x_i$  relative to the centre of mass of the  $N$  pseudoions leads to the Hamiltonian (2), with

$$\omega^2 \equiv \frac{\alpha N}{m}. \tag{5}$$

Since Hamiltonian (2) as well as Hamiltonian (1) does not include any spin interactions, we may factorize out the spin coordinates of the system and treat only the spatial coordinates. The model system described by equation (2) has been studied by several authors (Houston 1935, Bloch and Hsieh 1954, 1956 and Thompson 1967). However, all of these authors fail to take into consideration the fermion nature of the particles. Storer (1970) has extended the Thompson model, taking into account the fermion character of the particles. Employing an involved combinational calculation, he obtained the grand canonical partition function for the system. Because of the difficulties posed by the appearance of the chemical potential, the grand partition function was not calculated exactly and thus the thermodynamic properties of the system can only be obtained within an approximate scheme.

In spite of the apparent simplicity of the Hamiltonian (2) there is in fact no known method for obtaining an expression in closed form for the free energy. We can, however, utilize the results that we shall obtain for our model system given by Hamiltonian (1), in order to extract the free energy per particle corresponding to Hamiltonian (2). Specifically we shall perform the limit  $\lambda \rightarrow 1-0$  in such a way that the Fermi energy of  $H_{\lambda 0}$  be equated to the Fermi energy of  $H_0$  in equation (2). In § 4 we present an argument which leads us to claim that the free energy of the system described by equation (2) equals the free energy of the system defined by equation (1), when the limit is chosen in the manner just stated.

We shall now turn to the calculation of the free energy of the system represented by the Hamiltonian (1) using the double-time Green function technique. In its second-quantized form, the Hamiltonian (1) is

$$H_{\lambda\delta} = \hbar\Omega \sum_{n=0}^{\infty} n c_n^\dagger c_n - \frac{\hbar\omega^2}{4\Omega} \frac{\delta}{N} \left( \sum_{n=0}^{\infty} (n+1)^{1/2} (c_n^\dagger c_{n+1} - c_{n+1}^\dagger c_n) \right)^2 \quad (6)$$

where  $\Omega \equiv \omega(1-\lambda)^{1/2}$  and  $c_n^\dagger$  is the operator which creates a fermion in the harmonic oscillator energy level  $n$ . The  $c_n$  satisfy the usual anticommutation relations

$$\{c_n^\dagger, c_m\} = \delta_{n,m} \quad \{c_n^\dagger, c_m^\dagger\} = \{c_n, c_m\} = 0. \quad (7)$$

In the Hamiltonian (6), we have neglected the zero-point energy for the system. In terms of the operators

$$A \equiv \sum_{n=0}^{\infty} (n+1)^{1/2} (c_n^\dagger c_{n+1} - c_{n+1}^\dagger c_n) \quad (8)$$

$$B \equiv \sum_{n=0}^{\infty} (n+1)^{1/2} (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) \quad (9)$$

the Hamiltonian (6) can be written as

$$H_{\lambda\delta} = \hbar\Omega \sum_{n=0}^{\infty} n c_n^\dagger c_n - \frac{\hbar\omega^2 \delta}{4\Omega N} A^2 \quad (10)$$

where the operators  $A$  and  $B$  satisfy the commutation relations,

$$\begin{aligned} [A, H_{\lambda\delta}] &= \hbar\Omega B \\ [B, H_{\lambda\delta}] &= \hbar\Omega A + \frac{\hbar\omega^2 \delta}{\Omega} A \\ [B, A] &= -2N. \end{aligned} \quad (11)$$

To obtain the thermodynamic properties of the system, we calculate  $\langle H_1 \rangle$  the canonical ensemble average of the operator  $H_1$ . This thermal average is obtained by the method of double-time Green functions (Zubarev 1960). In general the correlation function of any two operators  $A$  and  $B$  is given by

$$\langle BA \rangle = \lim_{\epsilon \rightarrow 0} i \int_{-\infty}^{\infty} dE \frac{\langle\langle A; B \rangle\rangle_{E+i\epsilon} - \langle\langle A; B \rangle\rangle_{E-i\epsilon}}{e^{\beta E} - 1} \tag{12}$$

where  $\beta = (k_B T)^{-1}$  and  $\langle\langle A; B \rangle\rangle_E$  has the standard meaning of a Green function (Zubarev 1960).

For the Hamiltonian (10), we have

$$E \langle\langle A; A \rangle\rangle_E = \hbar \Omega \langle\langle B; A \rangle\rangle_E \tag{13}$$

$$E \langle\langle B; A \rangle\rangle_E = -\frac{2N}{2\pi} + \left( \hbar \Omega + \frac{\hbar \omega^2 \delta}{\Omega} \right) \langle\langle A; A \rangle\rangle_E. \tag{14}$$

Solving equations (12) and (14) for  $\langle\langle A; A \rangle\rangle_E$ , we obtain

$$\langle\langle A; A \rangle\rangle_E = -\frac{\hbar \Omega N}{2\pi \hbar \chi} \left( \frac{1}{E - \hbar \chi} - \frac{1}{E + \hbar \chi} \right) \tag{15}$$

where we have introduced  $\chi^2 \equiv \Omega^2 + \delta \omega^2$ . The thermal average of  $\langle A^2 \rangle$  is

$$\langle A^2 \rangle_\delta = -\Omega N (\Omega^2 + \delta \omega^2)^{-1/2} \coth(\frac{1}{2} \beta \hbar (\Omega^2 + \delta \omega^2)^{1/2}). \tag{16}$$

Thus,

$$\langle H_1 \rangle_\delta = \frac{1}{4} \hbar \omega^2 \delta (\Omega^2 + \delta \omega^2)^{-1/2} \coth(\frac{1}{2} \beta \hbar (\Omega^2 + \delta \omega^2)^{1/2}). \tag{17}$$

The canonical partition function can now be evaluated using (Kadanoff and Baym 1962, p 16)

$$[\ln Z]_{\delta=1} - [\ln Z]_{\delta=0} = -\beta \int_0^1 \frac{d\delta}{\delta} \langle H_1 \rangle_\delta \tag{18}$$

to obtain

$$[\ln Z]_{\delta=1} - [\ln Z]_{\delta=0} = \ln \sinh(\frac{1}{2} \beta \hbar (\Omega^2 + \omega^2)^{1/2}) - \ln \sinh(\frac{1}{2} \beta \hbar \Omega) \tag{19}$$

where  $[\ln Z]_{\delta=0}$  corresponds to the canonical partition function for a system of non-interacting fermions in a harmonic oscillator well.

In the thermodynamic limit, the free energy per particle is

$$\begin{aligned} f_{\lambda,1} - f_{\lambda,0} &= \lim_{N \rightarrow \infty} \{ -(\beta N)^{-1} ([\ln Z(\lambda)]_{\delta=1} - [\ln Z(\lambda)]_{\delta=0}) \} \\ &= \lim_{N \rightarrow \infty} \left\{ -(\beta N)^{-1} \ln \left( \frac{\sinh(\frac{1}{2} \beta \hbar \omega)}{\sinh(\frac{1}{2} \beta \hbar \omega (1 - \lambda)^{1/2})} \right) \right\}. \end{aligned} \tag{20}$$

As the right hand side of (20) is of order  $1/N$ , we see that the interaction part of the Hamiltonian does not contribute to the free energy per particle of the system. The thermodynamic properties of the system will be the same as those of a system of non-interacting fermions in a harmonic oscillator well.

### 3. The partition function in the canonical ensemble

The partition function for a system of fermions cannot in general be evaluated analytically because of the restrictions on the occupation numbers brought about by the Pauli exclusion principle. This difficulty exists even for the ideal Fermi gas. As a result it is customary to evaluate the grand partition function for the system and assume that in the thermodynamic limit, the same thermodynamic functions are obtained independent of the ensemble used for the calculation. In calculating the grand partition function, one must introduce the chemical potential  $\mu$  and use the requirement that the total number of particles be  $N$  in order to eliminate  $\mu$  from the thermodynamic functions. This however cannot be done analytically either for the ideal Fermi gas or for our system of noninteracting fermions in a harmonic oscillator well. We shall here calculate exactly and analytically the partition function for our system in the canonical ensemble. This is possible because the eigenvalues in our system are linear in  $n$ .

A study of the thermodynamic functions in the canonical ensemble for a many-fermion system has been performed previously by Brout and Englert (1960) and by Horowitz *et al* (1963). However, these authors restricted their attention to  $T = 0$  K. Furthermore, the thermodynamic functions were evaluated by a linked-cluster expansion, precluding the possibility of obtaining exact analytical expansions. By contrast, we obtain the thermodynamic function for our model in an exact and closed form.

The partition function in the canonical ensemble including the zero-point energy is given by

$$Z = \sum_{v_1} \sum_{v_2} \dots \sum_{v_N} \exp\left(-\beta \sum_j \{v_j \hbar \Omega(n_j + \frac{1}{2})\}\right) \quad (21)$$

where  $v_j = 0, 1$  is the occupation number of the  $j$ th energy level, and  $\hbar \Omega n_j$  is the energy of the  $j$ th level. In order to evaluate the sum in (21), we write

$$Z = (N!)^{-1} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_N=0}^{\infty} \exp\left(-\beta \hbar \Omega \sum_j n_j\right) \exp\left(-\frac{1}{2} \hbar \Omega N\right).$$

In terms of  $\epsilon \equiv \exp(-\beta \hbar \Omega)$  and  $\gamma \equiv \exp(-\frac{1}{2} \beta \hbar \Omega N)$  the partition function is given by

$$Z = \gamma (N!)^{-1} \sum_{n_1} \sum_{n_2} \dots \sum_{\substack{n_j \\ n_j \neq n_k}} (\epsilon)^{\sum n_j}.$$

Consider now a system of oscillators having the spectrum

$$\epsilon'_j = \hbar \Omega(n_j + \frac{3}{2})$$

and introduce

$$\sigma_N = (N!)^{-1} \sum_{n_1} \dots \sum_{\substack{n_j \\ n_j \neq n_k}} (\epsilon)^{\sum n_j}.$$

The canonical partition function corresponding to this system is given by

$$Z' = \gamma (N!)^{-1} \sum_{n_1} \dots \sum_{n_j} (\epsilon)^{\sum n_j} (\epsilon)^N = \gamma (\epsilon)^N \sigma_N.$$

Now, we introduce

$$\begin{aligned} \sigma' &= (N!)^{-1} \sum_{n_1} \dots \sum_{\substack{n_j \neq n_k \\ n_1 < n_2 < \dots \\ n_N}} (\epsilon)^{\sum n_j} \\ &= N!(N!)^{-1} \sum_{n_1 < n_2 < \dots} \dots \sum_{n_N} (\epsilon)^{\sum n_j} \\ &= \epsilon \sum_{\substack{n_2=2 \\ n_2 < n_3 \dots}} (\epsilon)^{\sum' n_j} + \epsilon^2 \sum_{\substack{n_2=3 \\ n_2 < n_3 \dots}} (\epsilon)^{\sum' n_j} + \dots + (\epsilon)^r \sum_{\substack{n_2=r+1 \\ n_2 < n_3 \dots}} (\epsilon)^{\sum' n_j} \end{aligned}$$

where the prime on the sum in the exponent indicates that the term  $j = 1$  is omitted. In general, however

$$\sum_{\substack{n_2=r+1 \\ n_2 < n_3 \dots}} (\epsilon)^{\sum' n_j} = (\epsilon)^{(r+1)(N-1)} \sigma_{N-1}.$$

Thus

$$\sigma'_N = \epsilon(\epsilon)^{2(N-1)} \sigma_{N-1} + \epsilon^2(\epsilon)^{3(N-1)} \sigma_{N-1} + \dots$$

but

$$\begin{aligned} \sigma'_N &= \epsilon^N \sigma_N \\ \sigma_N &= \epsilon^{-N} \sigma'_N = \sigma_{N-1} (\epsilon)^{N-1} (1 + \epsilon^N + \epsilon^{2N} + \dots) \\ &= \frac{\epsilon^{N-1}}{1 - \epsilon^N} \sigma_{N-1}. \end{aligned}$$

Solving this difference equation, we have

$$\sigma_N = \epsilon^{-N} \left( \prod_{j=1}^N (\epsilon^{-j} - 1) \right)^{-1}.$$

The partition function in the canonical ensemble is therefore given by

$$Z = \exp(\frac{1}{2} \beta \hbar \Omega N) \left( \prod_{j=1}^N \{ \exp(\beta \hbar \Omega j) - 1 \} \right)^{-1}. \tag{22}$$

The free energy for the system is then given by

$$\begin{aligned} F &= -\beta^{-1} \ln Z = -\frac{1}{2} \hbar \Omega N + \beta^{-1} \sum_{j=1}^N \ln(\exp(\beta \hbar \Omega j) - 1) \\ &= \frac{1}{2} \hbar \Omega N^2 + \beta^{-1} \sum_{j=1}^N \ln(1 - \exp(-\beta \hbar \Omega j)). \end{aligned} \tag{23}$$

Thus, the free energy per particle in the thermodynamic limit is

$$f = \lim_{N \rightarrow \infty} \frac{F}{N} = \lim_{N \rightarrow \infty} \left( \frac{1}{2} \hbar \Omega N + (\beta N)^{-1} \sum_{j=1}^N \ln(1 - \exp(-\beta \hbar \Omega j)) \right). \tag{24}$$

Expression (24) looks like the free energy per particle of a system of bosons having an energy spectrum  $\epsilon_n = \hbar \Omega n$ . We will show in a subsequent paper that indeed our system of fermions can be expressed in terms of its ‘associated boson field’, similar to the case of the Luttinger model (Luttinger 1963, Mattis and Lieb 1965).

From expression (24) it is clear that for an arbitrary  $\Omega$  which is of order unity, that is, independent of  $N$  or equivalently the size of the system, the system is nonextensive.

For example, the first term of (24) is of order  $N$ , giving the free energy of the system proportional to  $N^2$ . This result is not surprising and in fact is to be expected for a system of noninteracting fermions in a harmonic oscillator well. For the convenient case of  $T = 0$  K, because of the exclusion principle, the free energy is given by

$$F = \sum_{j=1}^N \hbar\Omega(j + \frac{1}{2}) = \frac{\hbar\Omega}{2}N^2 + \hbar\Omega N.$$

#### 4. Canonical partition function for the pseudoelectron system

In this section we shall study the thermodynamic properties of the system described by the Hamiltonian (2). This will be achieved by considering the system described by the Hamiltonian (1) for a particular form of the limit  $\lambda \rightarrow 1 - 0$ . For the latter system, as  $\lambda \rightarrow 1 - 0$ , the harmonic oscillator eigenfunctions extend over an ever increasing portion of space. However, as the system is enclosed in a box of length  $L$ , to insure that the walls of the box play no role, we shall require that

$$\omega(1 - \lambda)^{1/2} \gg \frac{\hbar}{mL^2}.$$

This restriction insures that at least the lowest eigenfunctions will be localized well within the box. Consistent with this requirement, we shall assume the functional form of the coupling constant to be

$$\omega(1 - \lambda)^{1/2} = \frac{J(c)}{N} \quad (25)$$

where  $J(c)$  is an arbitrary function of the particle density  $c = N/L$ . For  $\lambda$  given by (25), equation (20) now becomes

$$f_{\lambda \rightarrow 1,1} - f_{\lambda \rightarrow 1,0} = \lim_{N \rightarrow \infty} \frac{1}{\beta} \frac{\ln N}{N} \rightarrow 0. \quad (26)$$

That is, even for the special case of the limit  $\lambda \rightarrow 1 - 0$ , consistent with (25), we again obtain the basic result that the interaction part of the Hamiltonian (1) does not contribute to the free energy per particle.

For  $\lambda \rightarrow 1 - 0$  in accordance with (25), the right hand side of equation (24) is now of order unity and hence the system is extensive. Likewise, the system described by equation (2) is extensive, for the Hamiltonian describes a group of particles which are free except for the fact that the centre of mass oscillates harmonically with frequency  $\omega$ . (The oscillations of the centre of mass will contribute a term of order unity to the free energy (eg the energy of a one dimensional classical harmonic oscillator is  $k_B T$ , according to the equipartition theorem). Furthermore, the free energy of a group, of noninteracting particles is of order  $N$ . Hence the system is extensive.) The limit of  $\lambda \rightarrow 1 - 0$ , in accordance with (25), is still ambiguous for  $J(c)$  is as yet entirely arbitrary. However, we are interested in that choice of  $J(c)$  which insures that the limiting free energy of the system (1) will coincide with that of the system (2). A reasonable procedure for fixing  $J(c)$  is to demand that the Fermi energy of the system (2) be equal to the energy of the highest occupied level of system (1) at  $T = 0$  K. The Fermi energy of system (2) is  $\epsilon_F = (\pi\hbar c)^2/2m$  and the energy of the highest occupied oscillator level at  $T = 0$  K is  $N\hbar\omega(1 - \lambda)^{1/2} = \hbar J(c)$ .



Thus

$$J(c) = \frac{\pi^2 c^2 \hbar}{2m}. \tag{27}$$

We now study the free energy (24) in the limit  $\lambda \rightarrow 1 - 0$  in accordance with equations (25) and (27). Applying the Euler summation formula (Wylie 1960) to the sum in (24), we obtain

$$f = \epsilon_F \left\{ 1 - \tau^2 \left( \frac{\pi^2}{6} \right) + \tau^2 \sum_{n=1}^{\infty} n^{-2} e^{-n/\tau} \right\} \tag{28}$$

where  $\epsilon_F = (\pi \hbar c)^2 / 2m$  is the Fermi energy and  $\tau$  is the temperature in units of the Fermi temperature  $T_F$

$$\tau = \frac{T}{T_F} = \frac{2m}{(\pi \hbar c)^2 \beta}. \tag{29}$$

The details are presented in full in the Appendix. The infinite sum in (28) can be rewritten in terms of an integral using the identity

$$\sum_{n=1}^{\infty} n^{-2} e^{-n/\tau} = \int_0^{\infty} dx x \left\{ \exp \left( x + \frac{1}{\tau} \right) - 1 \right\}^{-1}.$$

It is clear that this function is analytic and its derivatives with respect to  $\tau$  exist to all order. Expressing  $f$  in terms of this integral makes it clear that derivatives of  $f$  with respect to  $\tau$  exist to all orders except for the physically uninteresting cases of  $\tau \rightarrow \infty$  and/or the density  $c \rightarrow 0$ . Since we are only interested in the degenerate quantum region, for our model system we shall only consider the region of  $0 \leq \tau \leq 1$ . Within this region we may conclude from expression (28) that our model system cannot exhibit any phase transition.

### 5. The thermodynamic functions—an exact calculation

In the previous section we have obtained expression (28) for the free energy per particle  $f$  for the pseudoelectron system. Our procedure for determining  $J(c)$  in (27) assumes that we are dealing with a degenerate quantum system. We will therefore restrict our discussion of the thermodynamic functions to the case of low temperatures, that is,  $\tau \ll 1$ .

In figure 1, we plot  $f$  as a function of  $\tau$ .

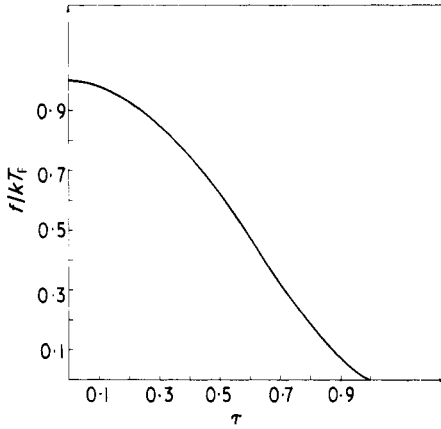
The pressure of the system is

$$p = c^2 \frac{\partial f}{\partial c} = c \epsilon_F \left( 1 + \frac{\pi^2}{3} \tau^2 - 2\tau^2 \sum_{n=1}^{\infty} n^{-2} e^{-n/\tau} - 2\tau \ln(1 - e^{-1/\tau}) \right). \tag{30}$$

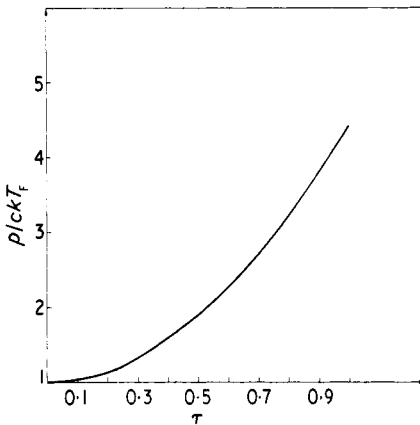
In figure 2, we plot the pressure as a function of  $\tau$ .

Using (28), we calculate the specific heat  $C_V$  for the system

$$C_V = -T \left( \frac{\partial^2 f}{\partial T^2} \right)_V = k_B \left( \frac{\pi^2}{3} \tau - 2\tau \sum_{n=1}^{\infty} n^{-2} e^{-n/\tau} + 2 \ln(1 - e^{-1/\tau}) + \tau^{-1} (1 - e^{1/\tau}) \right). \tag{31}$$



**Figure 1.** The free energy per particle  $f$  (in units of the Fermi energy,  $\epsilon_F$ ) as a function of the reduced temperature  $\tau = T/T_F$ .



**Figure 2.** The pressure  $p$  (in units of  $c\epsilon_F$ ) as a function of  $\tau = T/T_F$ .

Clearly, as  $\tau \rightarrow 0$  we have  $C_V \rightarrow 0$  guaranteeing that the third law of thermodynamics holds for our system. Figure 3 gives the specific heat as a function of  $\tau$ .

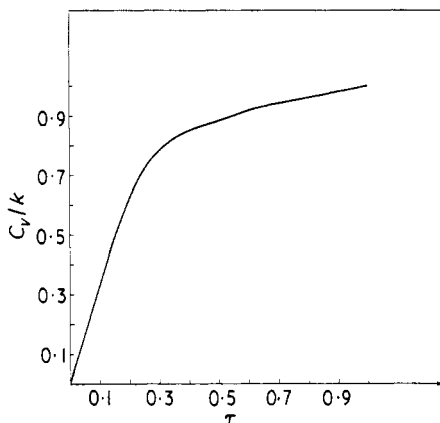
Using (30) we also calculated the isothermal compressibility  $K_T$  for our system

$$\frac{1}{K_T} = c\epsilon_F \left( 3 - \frac{\pi^2}{3}\tau^2 + 2\tau^2 \sum_{n=1}^{\infty} n^{-2} e^{-n/\tau} - 6\tau \ln(1 - e^{-1/\tau}) + \frac{4}{1 - e^{1/\tau}} \right). \quad (32)$$

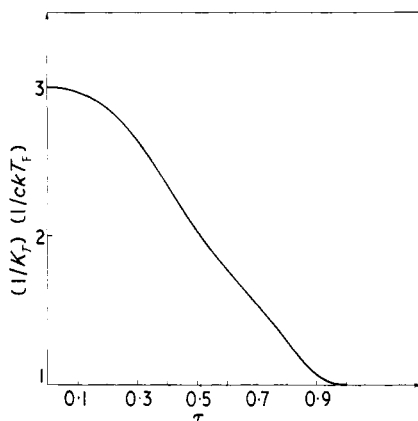
In figure 4, we plot  $K_T$  as a function of  $\tau$ .

### 6. Conclusion

We have studied a one dimensional model system of interacting fermions. We have shown that the model system is exactly soluble in the thermodynamic limit and we have evaluated the free energy of the system. It was shown that in the thermodynamic limit, the interaction term of the Hamiltonian does not contribute to the partition function of



**Figure 3.** The specific heat at constant volume  $C_V$  as a function of  $\tau = T/T_F$ .



**Figure 4.** The isothermal compressibility  $K_T$  (in units of  $c/kT_F$ ) as a function of  $\tau = T/T_F$ .

the canonical ensemble and the system is equivalent to a set of noninteracting fermions in an attractive harmonic oscillator well. As a special case, for a specific form of the coupling constant, the model Hamiltonian reduces to the Hamiltonian representing a particular two-species system of fermions interacting via pairwise harmonic forces. The thermodynamic properties of this system have been studied exactly. The pressure, the specific heat at constant volume, and the isothermal compressibility were all calculated in the canonical ensemble in closed form. These functions were shown to be analytic, proving that the system cannot possess a phase transition of any order.

### Acknowledgments

We wish to express our sincere thanks to Professors D C Mattis and G Horwitz for many helpful discussions and suggestions. One of us (EMW) also wishes to express her thanks to Professors M Luban and N Wisner for being a constant source of advice, encouragement and inspiration.

## Appendix

In this Appendix we evaluate the sum in (29) exactly, using the Euler summation formula. The Euler summation formula expresses a sum in terms of an integral and a remainder which may be expressed as an infinite sum

$$\sum_{i=a}^b f_i = \int_a^b f(x) dx + \frac{1}{2}(f(a) + f(b)) + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} (f_b^{(2j-1)} - f_a^{(2j-1)}).$$

We shall show however that the infinite sum vanishes for our function. Consider

$$I = \lim_{N \rightarrow \infty} \left( \frac{1}{N} \right) \sum_{j=1}^N \ln(\exp(\beta h \Omega j) - 1). \quad (\text{A.1})$$

In our case, using the Euler summation formula,

$$I = \lim_{N \rightarrow \infty} \left( \frac{1}{N} \right) \left( \int_1^N \ln(\exp(\beta h \Omega x) - 1) dx + \frac{1}{2} \ln(\exp(\beta h \Omega N) - 1) + \frac{1}{2} \ln(\exp(\beta h \Omega) - 1) + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} (f_N^{(2j-1)} - f_1^{(2j-1)}) \right). \quad (\text{A.2})$$

We shall first consider the infinite sum in (A.2). Let  $\beta h \Omega = \xi/N$

$$I = \lim_{N \rightarrow \infty} \left( \frac{1}{N} \right) \left( \int_1^N \ln(\exp(\xi x/N) - 1) dx + \frac{1}{2} \ln(e^{\xi} - 1) + \frac{1}{2} \ln(e^{\xi/N} - 1) + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} (f_N^{(2j-1)} - f_1^{(2j-1)}) \right)$$

$$f(x) = \ln(\exp(\xi x/N) - 1)$$

$$f^1(x) = -\frac{\xi}{N} \left\{ 1 - \exp\left(-\frac{\xi x}{N}\right) \right\}^{-1}$$

$$f^1(N) = -\frac{\xi}{N} (1 - e^{-\xi})^{-1}$$

$$f^1(1) = -\frac{\xi}{N} \left\{ 1 - \exp\left(-\frac{\xi}{N}\right) \right\}^{-1}$$

$$\lim_{N \rightarrow \infty} \left( \frac{1}{N} \right) f^1(N) = \lim_{N \rightarrow \infty} \left( -\frac{\xi}{N^2} \right) (1 - e^{-\xi})^{-1} \rightarrow 0.$$

Using equation (9.610; 1) (Gradshteyn and Ryzhik 1965)

$$f^1(1) = \sum_{n=0}^{\infty} \frac{B_n(-\xi/N)^n}{n!}$$

$$\lim_{N \rightarrow \infty} \left( \frac{1}{N} \right) f^1(1) = \lim_{N \rightarrow \infty} \left( \frac{1}{N} \right) \sum_{n=0}^{\infty} \frac{B_n(-\xi/N)^n}{n!} \rightarrow 0$$

$$f^2(x) = \left( \frac{\xi}{N} \right)^2 \exp\left(-\frac{\xi x}{N}\right) \left\{ 1 - \exp\left(-\frac{\xi x}{N}\right) \right\}^2$$

$$\begin{aligned}
 f^3(x) &= -\left(\frac{\xi}{N}\right)^3 \frac{1}{4} \frac{\cosh(\xi x/2N)}{\sinh^3(\xi x/2N)} \\
 \lim_{N \rightarrow \infty} \left(\frac{1}{N}\right) f^3(N) &= \lim_{N \rightarrow \infty} \left(-\frac{\xi}{N}\right)^3 \frac{1}{4N} \frac{\cosh(\frac{1}{2}\xi)}{\sinh^3(\frac{1}{2}\xi)} \rightarrow 0 \\
 \lim_{N \rightarrow \infty} \left(\frac{1}{N}\right) f^3(1) &= \lim_{N \rightarrow \infty} \left(-\frac{\xi}{N}\right)^3 \frac{1}{4N} \frac{\cosh(\xi/2N)}{\sinh^3(\xi/2N)} \\
 \sinh^3 x &= \left(x + \left(\frac{1}{3!}\right)x^3 + \left(\frac{1}{5!}\right)x^5 + \dots\right)^3 \\
 &= x^3 \left(1 + \left(\frac{1}{3!}\right)x^2 + \left(\frac{1}{5!}\right)x^4 + \dots\right)^3 \\
 \frac{(\xi/N)^2}{\sinh^3(\xi/2N)} &= \frac{(\xi/N)^3}{(\xi/2N)^3} \left\{1 + \left(\frac{1}{3!}\right)\left(\frac{\xi}{2N}\right)^2 + \dots\right\}^{-3} \\
 \lim_{N \rightarrow \infty} f^3(1) &= -2 \lim_{N \rightarrow \infty} \left(\frac{1}{N}\right) \left\{1 + \left(\frac{1}{3!}\right)\left(\frac{\xi}{2N}\right)^2 + \dots\right\}^{-3} \rightarrow 0.
 \end{aligned}$$

Clearly similar results are obtained for all odd derivatives of  $f(N)$  and  $f(1)$ . We therefore conclude that the last term in (A.2) does not contribute to  $I$ . Thus,

$$I = \lim_{N \rightarrow \infty} \left(\frac{1}{2N}\right) \ln(e^{\xi/N} - 1) + \lim_{N \rightarrow \infty} \left(\frac{1}{N}\right) \int_1^N \ln\left(\exp\left(\frac{\xi x}{N}\right) - 1\right) dx.$$

The evaluation of this integral leads to (30).

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